A NOVEL ROTATIONAL VLSI ARCHITECTURE BASED ON EXTENDED ELEMENTARY ANGLE SET CORDIC ALGORITHM

Cheng-Shing Wu and An-You Wu

Department of Electrical Engineering,
National Central University, Chung-Li 320, Taiwan, R.O.C.
E-mail:{benior, andyw]@ee.ncu.edu.tw

ABSTRACT

The CORDIC algorithm is a well-known iterative method for the computation of vector rotation. For applications that require forward rotation (or vector rotation) only, the angle recoding (AR) technique provides a relaxed approach to speed up the operation of the CORDIC algorithm. In this paper, we further apply the concept of AR technique to extend the elementary angle set in the micro-rotation phase. This technique is called Extended Elementary-Angle Set (EEAS) scheme. The proposed EEAS scheme provides a more flexible way in decomposing the target rotation angle in CORDIC operation, and its quantization error performance is better than AR technique. Meanwhile, we also propose an improved scaling operation, called Extended Type-II (ET-II) scaling operation, as the scaling scheme for the EEAS-based CORDIC algorithm. With the aid of the proposed EEAS scheme and ET-II scaling operation, we require only 39% iterations number in the iterative CORDIC structure, or use 39% hardware complexity in the parallel CORDIC structure compared with the conventional CORDIC approach. Hence, low-power/high-speed CORDIC VLSI architectures become feasible without sacrificing SQNR performance.

1. INTRODUCTION

The Coordinate Rotational Digital Computer (CORDIC) algorithm is a well-known iterative technique to perform various basic arithmetic operations [1]. The algorithm is very attractive for hardware implementation because it uses only elementary shift-and-add operations to perform the vector rotation in 2D plane. However, the major disadvantage of the CORDIC algorithm is its slow computational speed.

In [2], the Angle Recoding (AR) technique is proposed to solve the aforementioned problem. It is very suitable for applications that use CORDIC algorithm in only forward rotation mode (also known as vector rotation mode). Basically, the superior performance (improved angle precision and reduced iteration number) of the AR technique comes from the relaxation on the form of rotation sequence, $\mu(i)$, in the CORDIC algorithm. Motivated by this, we proposed an algorithmic-level improvement scheme, called Extended Elementary-Angle Set (EEAS) scheme. In addition to the relaxation on $\mu(i)$, the EEAS scheme further applies the relaxation on the elementary angles. In fact, the use of EEAS scheme has the effect of reducing the rotation angle error of the CORDIC algorithm in vector rotation mode. The reason is that we have more choices of elementary angles in approximating the target rotation angle.

Moreover, by applying similar idea of EEAS scheme, we propose an improved scaling operation, called Extended Type-II (ET-II) scaling operation. The ET-II scaling operation not only provides additional precision in the scaling process, but also inherits the features of EEAS scheme in the micro-rotation phase. This similarity of EEAS scheme and ET-II scaling operation makes the VLSI implementation reusable in iterative CORDIC structure, highly regular in parallel CORDIC structure.

With the aid of the proposed EEAS scheme and ET-II scaling operation, we can obtain additional 15dB SQNR gain compared with the conventional approach developed in [2] under the same complexity. We will show that given the same target SQNR performance, we require only 39% iterations number in the iterative CORDIC structure, or use 39% hardware complexity in the parallel CORDIC structure compared with the conventional CORDIC algorithm. Compared with the direct implementation (4 multipliers + 2 adders) with compatible error performance, we require only 28% hardware complexity. Hence, low-power/high-speed CORDIC VLSI architectures become feasible without sacrificing SQNR performance.

2. ANGLE RECODING TECHNIQUE

The CORDIC algorithm decomposes the rotation angle, $\theta$, into a combination of pre-defined elementary angles [1], i.e.,

$$\theta = \sum_{i=0}^{N-1} \mu(i)\pi(i) + \epsilon,$$

where $N$ is the number of elementary angles, $\mu(i) \in \{1, -1\}$ is the rotation sequence which determines the direction of the $i^{th}$ elementary angle of $a(i) = \tan^{-1}(2^{-i})$, and $\epsilon$ denotes the residue angle. Based on Eq. (1), the recurrence equations of the CORDIC algorithm can be written as

$$\begin{align*}
x(i+1) &= x(i) - \mu(i)y(i)2^{-i}, \\
y(i+1) &= y(i) + \mu(i)x(i)2^{-i},
\end{align*}$$

for $i = 0, 1, \ldots, N - 1$. Also, the final values, $x(N)$ and $y(N)$, need to be scaled by scaling factor, $P = \prod_{i=0}^{N-1} \sqrt{1 + 2^{-2i}}^{-1}$, to retain the norm of the initial vector $[x(0), y(0)]^T$. Other details of the conventional CORDIC algorithm can be found in [4].

From Eq. (2), we know that in the conventional CORDIC algorithm, each elementary angle needs to be performed sequentially so as to complete the micro-rotation phase. However, in the applications where the rotation angles are known in advance, it would be advantageous to relax the sequential constraint on the micro-rotation phase. The Angle Recoding (AR) technique is done by extending the set of $\mu(i)$ from $\{1, -1\}$ to $\{1, -1, 0\}$ [2]. With the relaxation on $\mu(i)$, for certain angles, we can obtain better approximation of $\theta$ (i.e., smaller residue angle, $\epsilon$) but with reduced iteration number. For example, consider the target angle $\theta = \pi/4$. The conventional CORDIC approach has to go through all the $N$ micro-rotations with $\mu(i)$ sequence $= \{1, -1, 1, 1, \ldots\}$. The residue angle, $\epsilon$, is $7.2 \times 10^{-5}$ for $N = 8$. However, with the AR technique, we can rotate $\theta = \pi/4$ by performing only one micro-rotation of elementary angle $\pi/0 = \arctan(2^0)$, and skipping all remaining micro-rotations. The resulting $\mu(i)$ sequence $= \{1, 0, 0, 0, 0, \ldots\}$, and it takes only one iteration with the residue angle $\epsilon = 0$.

3. OUR IMPROVED CORDIC ALGORITHM BASED ON EXTENDED ELEMENTARY-ANGLE SET (EEAS)

First, the AR technique presented in [2] imposes no restriction on the iteration number. As one can expect that rotation angles of different values may need unequal numbers of iterations, which may
lead to bus/timing alignment problems in VLSI circuits. In this paper, we employ one additional parameter, the maximum iteration number \( R_m \), to limit the number of iterations. By doing so, the total iterations number in the micro-rotation phase can be held fixed for various rotation angles \( \theta \). Now, the AR problem with fixed iterations number can be summarized as: Given a target angle \( \theta \) and the maximum iteration number, \( R_m \), find the rotation sequence \( \mu(i) \in \{1, -1, 0\} \) for \( 0 \leq i \leq N - 1 \), such that the residue angle error

\[
\xi_m = \theta - \sum_{i=0}^{N-1} [\mu(i) \cdot \tan^{-1}(2^{-a(i)})]
\]

is minimized subject to the constraint that the total iterations number

\[
\sum_{i=0}^{N-1} [\mu(i)] \leq R_m.
\]

### 3.1. Reformulation of the Angle Recoding Problem

Moreover, to facilitate the derivation of the proposed EEAS scheme, we first rewrite the AR problem described above in an alternative form as

\[
\xi_m = \theta - \sum_{j=0}^{R_m-1} \tan^{-1}(\alpha(j) \cdot 2^{-a(j)})
\]

where

- \( j, 0 \leq j \leq R_m - 1 \), denotes the iteration index,
- \( s(j) \in \{0, 1, \cdots, N - 1\} \) is the rotational sequence that determines the micro-rotation angle in the \( j^\text{th} \) iteration,
- \( \alpha(j) \in \{-1, 0, 1\} \) is the directional sequence that controls the direction of the \( j^\text{th} \) micro-rotation of \( \alpha(s(j)) \).

Eq. (5) shows that the reformulated AR problem is to find the combination of elements from a set, which consists of all possible value of \( \tan^{-1}(\alpha(j) \cdot 2^{-a(j)}) \), so that \( \xi_m \) can be minimized. We call such a set the elementary angles set (EAS) \( S_1 \), defined as

\[
S_1 = \{ \tan^{-1}(\alpha \cdot 2^{-a}) : \alpha \in \{-1, 0, 1\}, a \in \{0, 1, \cdots, N - 1\}\}
\]

The reason for using the subscript 1 will become apparent later. By doing so, the AR problem becomes: Given \( \theta \) and \( R_m \), find the combination of elementary angles from EAS \( S_1 \), such that the residue angle error \( \xi_m \) is minimized.

### 3.2. Extended Elementary-Angle Set (EEAS) Scheme

Next, we want to relax the constraint of elementary angles so as to extend the EAS \( S_1 \). By doing this, we can have more choices (elementary angles) in approximating the target angle \( \theta \). Hence, it is expected that the residue angle error \( \xi_m \) can be reduced correspondingly.

First, by observing Eq. (6), we can see that the EAS \( S_1 \) are comprised of arcsign of single signed-power-of-two (SPT) term, i.e., \( \tan^{-1}(\alpha \cdot 2^{-a}) \). In the problem of SPT-based digital filter design, one effective way to increase the coefficient resolution (hence the filter performance) is to employ more SPT terms to represent the filter coefficients [5]. Motivated by this, in our proposed relaxed scheme, we can easily extend the set by representing the elementary angles as the arcsign of the sum of two SPT terms. That is

\[
S_2 = \{ \tan^{-1}(\alpha_0 \cdot 2^{-a_0} + \alpha_1 \cdot 2^{-a_1}) : \alpha_0, \alpha_1 \in \{-1, 0, 1\}, a_0, a_1 \in \{0, 1, \cdots, N - 1\}\}
\]

We call it the Extended Elementary-Angle Set \( S_2 \) (EEAS \( S_2 \)). The subscript is used to denote the number of SPT terms.

To demonstrate the effectiveness of the proposed scheme, we first show the constellation of the elementary angles from \( S_1 \) and \( S_2 \) in Fig. 1 (a) and (b), respectively. As we can see, the number of reachable angles of \( S_2 \) is much larger than that of \( S_1 \). This implies that EEAS \( S_2 \) can yield smaller \( \xi_m \) than \( S_1 \) with a fixed number of micro-rotations of elementary angles.

![Figure 1: Constellation of elementary angles of (a) set \( S_1 \), (b) set \( S_2 \), with wordlength \( W = 8 \).](image)

### Table 1: Summary of the proposed EEAS-based CORDIC scheme.

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial vector</td>
<td>( [x(0), y(0)]^T )</td>
</tr>
</tbody>
</table>
| Micro-rotation phase | Given \( \beta = 0 \) to \( R_x-1 \) \[
\begin{align*}
\cos(\theta) & = x(0) \sin(\beta) + y(0) \cos(\beta) \\
\sin(\theta) & = x(0) \cos(\beta) - y(0) \sin(\beta)
\end{align*}
\]
| Scaling phase | \[
\begin{align*}
x & = \frac{x(0)}{\sqrt{x(0)^2 + y(0)^2}} \\
y & = \frac{y(0)}{\sqrt{x(0)^2 + y(0)^2}}
\end{align*}
\]

Based on the EEAS \( S_2 \) developed in Eq. (7), the recurrence equations of the conventional CORDIC algorithm in Eq. (2) can be modified as

\[
\begin{align*}
x(j+1) &= x(j) - \alpha(j) \cdot 2^{-a(j)} + \alpha_1(j) \cdot 2^{-a_1(j)} \\
y(j+1) &= y(j) + \alpha_0(j) \cdot 2^{-a_0(j)} + \alpha_1(j) \cdot 2^{-a_1(j)}
\end{align*}
\]

for \( 0 \leq j \leq R_m - 1 \), where \( \alpha_0(j), \alpha_1(j), s_0(j), s_1(j) \) denote the parameters to control the \( j^\text{th} \) micro-rotation of elementary angle \( \tan^{-1}(\alpha_0(j) \cdot 2^{-a_0(j)} + \alpha_1(j) \cdot 2^{-a_1(j)}) \). In Table 1, we summarize the micro-rotation procedure as well as the scaling operation of the proposed EEAS-based CORDIC scheme.

### 4. SEARCHING ALGORITHMS OF THE EEAS SCHEME

With the newly derived EEAS \( S_2 \), the optimization problem of AR scheme becomes: Given \( \theta \) and \( R_m \), find the \( \alpha_0(j), \alpha_1(j), s_0(j), s_1(j) \) (i.e., the combination of elementary angles from EEAS \( S_2 \)), such that the residue angle error

\[
\xi_m = \theta - \sum_{j=0}^{R_m-1} \tan^{-1}(\alpha_0(j) \cdot 2^{-a_0(j)} + \alpha_1(j) \cdot 2^{-a_1(j)})
\]

can be minimized.

One intuitive solution to the optimization problem is to perform exhaustive search, i.e., exploring all possible combinations of \( E(j) \). However, it is practically impossible due to its extraordinarily heavy computational complexity.

Instead, the optimization problem of \( \xi_m \) in Eq. (9) can be solved by the Greedy Algorithm (GA). In [2], similar approach has been used to solve the conventional AR problem described in Section 2. The greedy algorithm tries to approximate the remaining angle using a closest elementary angle in each search step without looking ahead of future steps. Then, by successively applying such an operation, the accumulated angle can continuously approach the target angle \( \theta \) until the searching algorithm is terminated; it terminates when no further improvement can be found, or the \( R_m \)th micro-rotation angle is determined.
5. EXTENDED TYPE-II SCALING OPERATION

From the modified recurrence equations in Eq. (8), each micro-rotation enlarges the norm of the vector by a factor of
\[ k(i) = \sqrt{1 + (\alpha(i) \cdot 2^{-s(i)} + \alpha(i) \cdot 2^{+s(i)})^2}, \]
for \( i = 0, 1, \ldots, R_a - 1 \). Hence, after the completion of Eq. (8), the accumulated norm equals
\[ k = \prod_{i=0}^{R_a-1} \sqrt{1 + (\alpha(i) \cdot 2^{-s(i)} + \alpha(i) \cdot 2^{+s(i)})^2}. \]

To ensure the preservation of the norm of the input vector, we have to multiply the vector of \([x(R_a), y(R_a)]^T\) by a scaling factor of \( P = 1/k \) after the sequence of micro-rotations, as summarized in Table 1.

5.1. Conventional Scaling Approach

To save hardware complexity, practical implementation performs the scaling operation by quantizing the scaling factor, \( P \), in the following two forms:

Type I: \[ \hat{P} = \sum_{m=0}^{R_a-1} k(m) \cdot 2^{-q(m)}, \]
Type II: \[ P = \prod_{m=0}^{R_a-1} [1 + k(m) \cdot 2^{-e(m)}], \]

where \( \hat{P} \) is the quantized value of \( P \), \( R_a \) is the maximum iteration number in the scaling phase, \( k(m) \in \{1, -1, 0\} \), and \( q(m) \in \{0, 1, \ldots, W - 1\} \). By doing so, we can approximate the multiplication of \( P \) with only \( R_a \) shift-and-add operations, which eliminates the significant overhead of scaling multipliers. As one can expect, this approximation process will introduce some quantization noise, and the noise increases as \( R_a \) decreases. As with the micro-rotation phase, we introduce another performance index of scaling approximation error \( \xi_s \), which is defined as
\[ \xi_s = \left| P - \hat{P} \right|, \]

to describe the amount of error introduced by the approximation process in the scaling phase.

5.2. The Extended Type-II (ET-II) Scaling Operation

In this paper, we also propose an improved scaling operation for the ESAS-based CORDIC algorithm. Similar to the ESAS scheme, the basic idea is to increase the number of possible values that can be represented by \( [1 + k(m) \cdot 2^{-e(m)}] \). This can be achieved by employing one extra SPT term in Eq. (12). Then, we obtain
\[ \hat{P} = \prod_{m=0}^{R_a-1} [1 + k(m) \cdot 2^{-e(m)} + k(m) \cdot 2^{+e(m)}]. \]

Since the relaxation of scaling procedure is done by extending the conventional Type-II scaling operation, we call it Extended Type-II (ET-II) scaling operation. By doing so, it is expected that we can obtain more accurate approximation of \( P \) due to the increased design parameters.

By quantizing \( P \) in this way, the scaling operation can be accomplished within \( R_a \) iterations by using the recurrence equations:
\[ x(m+1) = x(m) + k(m) \cdot 2^{-e(m)} + k(m) \cdot 2^{+e(m)} \cdot \hat{x}(m) \]
\[ y(m+1) = y(m) + k(m) \cdot 2^{-e(m)} + k(m) \cdot 2^{+e(m)} \cdot \hat{y}(m) \]

The initial settings of scaling phase are set as \( \hat{x}(0) = x(R_a) \) and \( \hat{y}(0) = y(R_a) \).

The advantages of the proposed ET-II scaling operation are: 1) Eq. (14) adds one more design parameter compared with conventional Type-I and Type-II scaling operations. The flexibility helps to reduce the approximation error, \( \xi_s \). 2) The recurrence equations of Eq. (15) are very similar to the ESAS scheme in the micro-rotation phase, as shown in Eq. (8). This suggests that the ET-II scaling operation can share the same circuits with micro-rotation module. The consistency of the circuits in both phases makes the structure more regular, which is a desirable feature in VLSI implementation.

5.3. Searching Algorithm of the Scaling Operation

By substituting Eq. (14) into Eq. (13), we have the scaling approximation error, \( \xi_s \), as
\[ \xi_s = \left| P - \prod_{m=0}^{R_a-1} [1 + k(m) \cdot 2^{-e(m)} + k(m) \cdot 2^{+e(m)}] \right|. \]

Now the problem is to determine the scaling parameters of \( k(m) \), \( k_1(m) \), and \( q(m) \) for \( 0 \leq m \leq R_a - 1 \), such that the scaling approximation error, \( \xi_s \), can be minimized.

Comparing the optimization problems described by Eq. (9) and Eq. (16), it is interesting to see that these two optimization problems (one in the micro-rotation phase, and the other one in the scaling phase) have some common features: 1) Both of them try to minimize a target value that is known in advance. 2) The parameter sets have the same form of “Sum of Two SPT terms”, as shown on the righthand side of Eq. (9) and Eq. (16). 3) These two forms of “Sum of Two SPT terms” are multiplied and summed over their respective indices, and are not dependent on each other. Since the performance index \( \xi_s \) is calculated only for the micro-rotation phase, we can minimize \( \xi_s \) by choosing the most suitable \( k(m) \) and \( q(m) \) for minimizing \( \xi_s \).

Based on the observations, we can conclude that the problem of minimizing \( \xi_s \) is similar to that of \( \xi_s \), in the micro-rotation phase. It suggests that the Greedy algorithm can also be applied to the optimization problem of Eq. (16).

6. SIMULATION OF ESAS-BASED CORDIC ALGORITHM

Now we combine the micro-rotation phase and the scaling phase, which finishes the complete ESAS-based CORDIC rotation. In this simulation, two kinds of combination are performed:

- Combination 1: The AR technique in the micro-rotation phase
- Combination 2 (the proposed approach): The ESAS scheme in the micro-rotation phase (use EAS S2 of Eq. (6)) and Type-II scaling operation in the scaling phase.

In Fig. 2, the SQNR performance of these two combinations is plotted for 64 uniform spaced angles in the region from 0 to \( \pi/4 \), i.e., \( \theta = 0, \pi/32, \pi/16, \pi/8, \cdots, \pi/2 \). Actually, these angles are the basis rotation angles (the twiddle factors) of the 512-point FFT/IFFT. For each rotation angle \( \theta \), the SQNR value is obtained by first computing the residue error, \( \xi_s \), and then the approximation error, \( \xi_s \), in the two separate phase. Then, using the equation of
\[ SQNR(\text{dB}) = 10 \cdot \log_{10} \left( \frac{1}{\xi_s^2 + \xi_s^2} \right), \]
we can accurately estimate the SQNR performance.

The results in Fig. 2 indicate: 1) The proposed approach consistently behaves better than the AR approach for all the rotation angles. The ensemble-averaged SQNR value of the 65 angles is 84.86dB, which is about 15dB greater than that of conventional AR approach. 2) The proposed approach provides apparent SQNR improvement for those large rotation angles (\( \theta > 6\pi/32 \)). 3) In certain applications, what we concerned is the worst case instead of the ensemble-averaged performance. In this simulation, the worst case generated by our approach is 57.6dB, while the one of the conventional AR approach is as low as 45.3dB. The SQNR difference is about 12.3dB.

\footnote{The rotation angles, which are greater than \( \pi/4 \), can be easily performed by introducing the pre-rotation angle \( \pi/8 \).}
7. VLSI ARCHITECTURE OF THE EEAS-BASEDCORDIC ALGORITHM

In Fig. 3, we illustrate the iterative structure for the proposed EEAS-based CORDIC algorithm. It is built based on the conventional iterative CORDIC structure in [4]. It consists of 2 barrel shifter (BS), 4 multiplexers (MUX) and 4 adders/subtractors. As shown in Fig. 3, two separate phases are performed to complete single CORDIC rotation, i.e., the micro-rotational phase (marked by solid line) and the scaling phase (marked by dash line). In each phase, three kinds of control signal are used to control the operations:

- $s_0(i)$ and $s_1(i)$ in micro-rotation phase as well as $q_0(m)$, $q_1(m)$ in scaling phase: they control the number of bits to be shifted by barrel shifters.
- $a_0(i)$ and $a_1(i)$ in micro-rotation phase as well as $b_0(m)$, $b_1(m)$ in scaling phase: they determine the operations of adders/subtractors.
- Control signal, $C$: it governs the phase switching of the iterative structure.

All the control signals can be generated by the Greedy algorithm in advance.

By unfolding the iterative implementation of Fig. 3, we can obtain the parallel structure as depicted in Fig. 4. The structure is composed of $(R_{w} + R_{r})$ basic EEAS-based CORDIC processors connected in cascade form, in which the $R_{w}$ leading processors perform the micro-rotations and the following $R_{r}$ processors execute the scaling operations. Each basic processor performs one iteration as specified in Fig. 3. Moreover, for the case that the parallel structure is dedicated to perform a particular rotation angle, the operation of each processor is kept fixed. We can thus save the hardware complexity easily by replacing all the control circuits, barrel shifters, and multiplexers with only wire routing.

In Table 2, several existing approaches/ algorithms performing vector rotation in 2D plane are compared. Note that the simulation results are ensemble-average of 1000 different rotation angles. As we can see that by using the proposed EEAS-based CORDIC algorithm, vector rotation can be accomplished with only 16 adders/ subtractors of 16-bit wordlength, while the SQNR performance, in an average sense, is as high as 84.8dB. Compared with the direct implementation with compatible error performance, which demands four 16-bit*14-bit array multipliers and two 16-bit adders, we require only 28% hardware complexity. Compared with conventional CORDIC algorithm and the AR technique, only 29% and 80% hardware complexity is required, respectively.

8. CONCLUSION

In this paper, motivated by the SPT representation of elementary angles, we proposed two novel schemes. The EEAS scheme improves the error performance in the micro-rotation phase. Then, the ET-II scaling operation deals with the scaling phase of EEAS-based CORDIC algorithm. Putting these techniques together, vector rotation can be easily accomplished with a few shift-and-add operations without sacrificing error performance. The significant improvement makes applications, which call for high complexity, feasible, such as high-point/high-speed discrete transformations (FFT, DCT) and high-order digital lattice filters.

References