High-Convergence-Speed Low-Computation-Complexity
SVD Algorithm for MIMO-OFDM Systems
Graduate Institute of Electronics Engineering, National Taiwan University, Taipei, Taiwan

ABSTRACT

Multiple-input multiple-output (MIMO) wireless communication systems with orthogonal frequency-division multiplexing (OFDM) achieve high spectral efficiency high channel capacity, and many MIMO-OFDM systems use the spatial multiplexing technique to improve the system performance. The MIMO-OFDM systems require the singular values and the corresponding singular vectors of the channel matrix by exploiting singular value decomposition (SVD). The information of the right singular-vector matrix can be fed back to the transmitter for linear precoding to improve the error performance when facing the channel matrix with rank deficiency problem. This work proposes a SVD algorithm with fast convergence speed, which is suitable for the MIMO channels with short coherent time. The proposed SVD algorithm has the following features: (1) low total computational complexity, (2) fast convergence speed, (3) the ability of reconfigurable to different numbers of transmit and receiver antennas, and (4) insensitive to the dynamic range problem, which is suitable for hardware implementation.

I. INTRODUCTION

The need for high-throughput communication is increasing due to the demand of high-quality wireless transmission such as IEEE 802.11n WLAN systems and IEEE 802.16e WiMAX systems. We can use antenna arrays at both transmitter and receiver to construct multiple-input multiple-output (MIMO) transceivers. Using the orthogonal frequency-division multiplexing (OFDM) technique, we can significantly enhance the throughput by channel diversity and array gain. In MIMO-OFDM wireless systems, spatial multiplexing is a common technique used for the antennas to increase the diversity against multi-path fading or spatially separate devices. When encountering the channel matrix with the rank deficiency or high correlation between the antennas, the spatial multiplexing technique suffers from the degradation of the bit-error-rate (BER) performance. One effective way to increase the robustness of spatial multiplexing to an ill-conditioned channel matrix is to feedback the proper linear precoding matrix derived by the channel state information (CSI) to the transmitter. Recently, signals can be separated even from closely spaced transmit antennas in richly scattering channels using adaptive array combining techniques. Consequently, the use of multiple antennas at both transmitter and receiver could increase the throughput linearly with the number of transmit or receive antennas.

The right singular vector matrix derived from the SVD results is the optimal precoding matrix for linear receivers such as zero-forcing (ZF) minimum mean square error (MMSE) receivers [1]. There have been researches about the singular value decomposition (SVD) algorithms for MIMO-OFDM applications. An algorithm of updating the singular vectors of the channel matrix by periodic pre- and post-multiplication by Jacobi rotation matrices was proposed in [2] with high computational complexity. [3], [4] and [5] proposed adaptive SVD algorithms for MIMO applications without CSI, but their convergence time requires hundreds of samples per carrier. Hence, these algorithms are not suitable for the MIMO channel whose coherent time is short.

In this work, we propose a fast convergence speed SVD algorithm, which need only 2–4 iterations in average to obtain each singular vector. By applying the proposed adaptive binary shift mechanism, we can ensure that the range of the variables during iterative computation will not grow exponentially. The mechanism almost has no harm to the convergence speed and the final results due to the constrained precision. This algorithm is also reconfigurable to different sets of transmit and receive antenna pairs, which is suitable to modern MIMO-OFDM applications.

The paper is organized as follows. The system model is described in section II, and the details of the operation and analysis of the proposed SVD algorithm are presented in section III. In section IV, the simulation results are presented.

II. SYSTEM MODEL

Consider a wireless MIMO system in a frequency nonselective, slowly fading channel. Suppose $N_t$ and $N_r$ antennas are used at the transmitter and receiver. The equivalent channel model is given by

$$r = Hs + n,$$  (1)

where $H \in \mathbb{C}^{N_r \times N_t}$ is the complex channel matrix with the $(p,q)$-th element which is the random fading between the $p$-th receive and $q$-th transmit antennas. $n \in \mathbb{C}^{N_r}$ is the additive noise source and is modeled as a zero-mean, circularly symmetric, complex Gaussian random vector with statistically independent elements. The $p$-th element of $s \in \mathbb{C}^{N_t}$ is the symbol transmitted at the $p$-th transmit antenna, and that of $r \in \mathbb{C}^{N_r}$ is the symbol received at the $p$-th received antenna.

After getting the CSI, we can decompose the channel matrix $H$ into the following equation,

$$H = USV^H,$$  (2)

where $U$ is an $N_t \times N_t$ unitary matrix, $V$ is an $N_r \times N_r$ unitary matrix, and $\Sigma$ is a matrix with only nonzero main diagonal entries which are the nonnegative square roots of the eigenvalues of $H^H H$. We define the diagonal elements of $\Sigma$ are in descending order,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}, \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r.$$  (3)

$u_i$'s and $v_i$'s are the corresponding left and right singular vectors, which means

$$U = [u_1 \; u_2 \; \cdots \; u_{N_t}],$$  (4)

$$V = [v_1 \; v_2 \; \cdots \; v_{N_r}].$$  (5)

In the following exposition, we assume that each antenna has equal power allocation. Substituting the SVD of the matrix $H$, (1) becomes
where $\mathbf{n}' = \mathbf{U}^H \mathbf{n}$. Noticing that the distribution of $\mathbf{n}'$ is invariant under unitary transformation, which means the multiplication of the AWGN by a unitary matrix does not cause any noise enhancement. The multiantenna channel is equivalent to $\min(N_s, N_r)$ independent parallel Gaussian subchannels at most. Each subchannel has gain which is the singular value of the fading matrix $\mathbf{H}$.

III. THE PROPOSED SINGULAR VALUE DECOMPOSITION ALGORITHM

A. The Concept of the Proposed SVD Algorithm

The basic idea of the proposed SVD algorithm is not to compute the decomposition directly, but derive the direction of the right singular vectors by iterative computation. We speed up the decomposition directly, but derive the direction of the right singular vectors by iterative computation. We speed up the convergence speed by using the matrix multiplications iteratively. At the same time, we apply the proposed adaptive binary shift mechanism to prevent the growth of the variable range. Unlike the traditional power iteration method [6, p.330], this work provides fast convergence speed of getting the results of SVD, and needs only 8–10 bits for the variables during the iterative computation in our simulations.

To simplify the SVD problem from three unknown matrices, $\mathbf{U}, \Sigma$, and $\mathbf{V}$, to two unknown matrices, we then define the matrix $\mathbf{P}_1^{(n)}$,

$$
\mathbf{P}_1^{(n)} = \mathbf{H}^H = \mathbf{V} \Sigma^2 \mathbf{V}^H = \sum_{i=1}^{N_r} \sigma_i^2 \mathbf{v}_i \mathbf{v}_i^H. 
$$

where $\mathbf{P}_1^{(n)}$ indicates the updating matrix in the $n$-th iteration of the proposed algorithm for deriving the $k$-th singular vector $\mathbf{v}_k$, such that $\mathbf{P}_1^{(1)} = \mathbf{H}^H$ at the first step. The value of the maximum iteration number, $n$, can be defined in advance. We only have to solve two unknown matrices, $\Sigma$ and $\mathbf{V}$ in (8).

By doing $n$ times self-multiplication on $\mathbf{P}_1$, we then get

$$
\mathbf{P}_1^{(n+1)} = \mathbf{P}_1^{(n)} \times \mathbf{P}_1^{(n)} = \mathbf{V} \Sigma^{2n} \mathbf{V}^H = \sum_{i=1}^{N_r} \sigma_i^{2n} \mathbf{v}_i \mathbf{v}_i^H. 
$$

where $n$ means the $n$-th iterative multiplication. The gap between the component, $\sigma_i^{2n} \mathbf{v}_i \mathbf{v}_i^H$, corresponding to the greatest singular value, $\sigma_1$, is significantly amplified. In other words, the gap between $\sigma_i^{2n} \mathbf{v}_i \mathbf{v}_i^H$ and other components, $[\sigma_j^{2n} \mathbf{v}_j \mathbf{v}_j^H, j \neq 1]$, corresponding to smaller singular values is enlarged after each iterative matrix multiplication. The convergence speed is proportional to $O \left( \frac{2^n}{\sigma_2^{2n}} \right)^{\frac{1}{2}}$. That is to say, the proposed algorithm has fast convergence speed. If $n$ is greater than four in this algorithm, we can have the following approximation

$$
\mathbf{P}_1^{(n)} = \mathbf{P}_1^{(1)} = \sum_{i=1}^{N_r} \sigma_i^{2n} \mathbf{v}_i \mathbf{v}_i^H \approx \sigma_1^{2n} \mathbf{v}_1 \mathbf{v}_1^H. 
$$

The first column vector of $\mathbf{P}_1^{(n)}$, says $\mathbf{p}_{1,1}$, can be normalized to be $\mathbf{p}_{n,1}$ such that

$$
\mathbf{v}_1 = \mathbf{p}_{n,1} \approx e^{j\phi} \mathbf{v}_1. 
$$

where $\mathbf{v}_1$ is the estimated unit vector of $\mathbf{v}_1$ and $\phi$ is the phase difference between $\mathbf{v}_1$ and $\mathbf{v}_1$. Exploiting the relation in (11), we can rewrite (10) to be the following equation,

$$
P_1^{(n)} \approx \sigma_1^{2n} \mathbf{v}_1 \mathbf{v}_1^H = \sigma_1^{2n} (e^{-j\phi} \mathbf{v}_1)(e^{-j\phi} \mathbf{v}_1)^H = \sigma_1^{2n} \mathbf{I},
$$

which shows that $\mathbf{v}_1$ has the same property to fulfill the condition in (9). In other words, any $\mathbf{v}_k$ with only phase difference from $\mathbf{v}_1$ can be treated as a correct estimation of $\mathbf{v}_1$.

B. The Proposed Adaptive Binary Shift Mechanism

The column vector $\mathbf{p}_{n,1}$ is normalized to get $\mathbf{v}_1$ at last no matter what magnitude of $\mathbf{p}_{n,1}$ is. So we can do re-scaling after each self-multiplication in (9). That is, the equation (10) can be modified as

$$
P_1^{(n)} = \left( \prod_{i=1}^{n} k_i \right) \sum_{i=1}^{N_r} \sigma_i^{2n} \mathbf{v}_i \mathbf{v}_i^H = \left( \prod_{i=1}^{n} k_i \right) \sigma_1^{2n} \mathbf{v}_1 \mathbf{v}_1^H, 
$$

where $k_i$ is the re-scaling factor in each iteration. Considering the cost-efficient hardware implementation, we choose $k_1$ to be the power of two so that only binary shifts needed in the re-scaling operation. At the beginning of the each iteration, let the re-scaling factor, $k_1$, be $2^{k_1}$, (13) can be formed into

$$
P_1^{(n)} = \left( \prod_{i=1}^{n} 2^{k_i} \right) \sum_{i=1}^{N_r} \sigma_i^{2n} \mathbf{v}_i \mathbf{v}_i^H = \left( 2^{k_1} \right) \sigma_1^{2n} \mathbf{v}_1 \mathbf{v}_1^H. 
$$

The optimal $k_i$ chosen in the $n$-th iteration of deriving the $k$-th singular vector is adaptively set to be

$$
k_i = \left\lfloor 2^{k_1 - m} \max (abs (\mathbf{P}_1^{(n-1)}))_{p,q} < 2^{k_1} \right\rfloor, 
$$

where $abs()_{m,n}$ means the absolute value of the ($p,q$)-th entry of the indicated matrix.

In the proposed SVD algorithm, we first derive the unit singular column vectors, $\mathbf{v}_k$, of $\mathbf{V}$ instead of finding out correct $\sigma$. We only have to keep the direction of the singular vectors during iterations, which makes us be free from the problem of dynamic range caused by the explosively growing $\sigma_i$ during the iterative self-multiplications. This property is suitable for the cost-effective hardware implementation.

C. De-correlation

After finding out $\mathbf{v}_1$, the correlated components of $\mathbf{v}_i$ in $\mathbf{P}_1^{(1)}$ should be eliminated for deriving the next estimated singular vector, $\mathbf{v}_2$. The singular vectors, $\mathbf{v}_i$’s, have two properties as following equations,

summation property : $\sum_{i=1}^{N_r} \mathbf{v}_i \mathbf{v}_i^H = \mathbf{I}_{N_r}$, (16)

and orthogonal property : $\mathbf{v}_i^H \mathbf{v}_j = 0, \forall i \neq j$, (17)

where $\mathbf{I}_{N_r}$ is an $N_r$-by-$N_r$ identity matrix. By exploiting the two properties, the de-correlation operation can be executed as
Hardware needed at the same time. The computation of the binary shift mechanism so as to reduce the critical path and the required in the iterative multiplication due to the proposed adaptive norm of each column in the estimated left singular vectors, respectively. By computing the rooters, and dividers are required for the derivation of singular vectors. As a conclusion, the proposed multiplication of getting the last right singular vector, be written as operation. Without losing generality, the de-correlation operation can be applied to any smaller complex channel matrices than with almost no hardware overhead.

Since we have only $\text{CMACs}$ for the matrix $\mathbf{H}$ to be the basic computational complexity unit. For all the same iterative operations are executed to get the results of SVD. After the matrix is unneeded, and it reduces the total computational complexity. Something should be noticed that the main computations and constraint of wordlength precision at the same time. The detailed steps will be listed as follows, and not to exceed the constraint of wordlength precision at the same time. The computation of $\tilde{\Sigma}$ requires no iterative process and is outside the loop, which means we can use greater wordlength for the computation of $\tilde{\Sigma}$ and $\tilde{\mathbf{U}}$ to get the better performance without increasing much hardware overhead.

D. Reconfigurability and Complexity Comparison

In our proposed SVD algorithm, we can handle an $N_t$-by-$N_r$ complex channel matrix, $\mathbf{H}$. In the situation of the MIMO channel with less transmit or receive antennas, we will then get an $N_t'$-by-$N_r'$ channel matrix, $\mathbf{H}'$, where $N_t' \leq N_t$ and $N_r' \leq N_r$. All we have to do is just put matrix $\mathbf{H}'$ into the first $N_t'$-by-$N_r'$ entries of the original storage space of $\mathbf{H}$, and fill all other unused entries with zero’s then the same iterative operations are executed to get the results of SVD. Since we have only $N_r'$ right singular vectors to derive, we can terminate the iterative process after getting the first $N_r'$-1 singular vectors and start to do the MGS operation to get the remaining singular vectors and singular values. As a conclusion, the proposed SVD algorithm can be easily applied to any smaller complex channel matrices than with almost no hardware overhead.

Assume that the maximum iteration number needed for each singular vectors is $n_{\text{max}}$ and use one complex multiplier and adder (CMAC) to be the basic computational complexity unit. For all the iterations we need about $n_{\text{max}} \left( N_r N_t \right)$ CMACs for the matrix multiplication of the Hermitian matrix $\mathbf{P}_k^{(1)}$ in addition, we still need $2N_t(N_t-1)$ and $N_t^2(N_t-2)$ CMACs for MGS and de-correlation operation, respectively. At last, $(N_r \times N_t)$ CMACs, few square rooters, and dividers are required for the derivation of $\tilde{\mathbf{U}}$ and $\tilde{\Sigma}$. The number of $n_{\text{max}}$ set in our algorithm is about 4 at most, which will be shown in section IV. Though the adaptive SVD algorithm proposed in [3] and [4, 5] have less complexity in each iteration and be able to operate without CSI, but they need about 400 and (500x $N_r$) samples to converge, respectively. The total computational complexity of our algorithm is much lower. The total computational complexity comparison is shown in Table I.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>3 x 3 channel</th>
<th>4 x 4 channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVD algorithm in [3]</td>
<td>18000 CMACs</td>
<td>32000 CMACs</td>
</tr>
<tr>
<td>SVD algorithm in [4, 5]</td>
<td>15600 CMACs</td>
<td>20400 CMACs</td>
</tr>
<tr>
<td>Proposed SVD algorithm</td>
<td>111 CMACs</td>
<td>464 CMACs</td>
</tr>
</tbody>
</table>

E. Algorithm Flow and the Architecture

Fig. 1 shows the flow chart of the proposed SVD algorithm in this work. The detailed steps will be listed as follows,

Step 1 : Get the complex channel matrix $\mathbf{H}$.
Step 2 : Derive the updating matrix $\mathbf{P}_k^{(1)}$ of the right singular vector corresponding k-th singular value.
Step 3 : Do the matrix multiplication and adaptive binary shift to approach the desired singular vector and not to exceed the constraint of wordlength precision at the same time.
Step 4 : Check if the set maximum iteration number is reached or not. Go back to step 3 if the condition is not satisfied, and go to step 5 if yes.
Step 5 : Do the de-correlation operation and check if all singular vectors are solved or not. Go back to step 2 if not, and go to step 6 if the condition is satisfied.
Step 6 : Derive the results of $\tilde{\mathbf{U}}, \tilde{\Sigma}$, and $\tilde{\mathbf{V}}$.

![Figure 1](image-url)
error (NMSE), which is defined as
\[
\text{NMSE} = \frac{\|\bar{Y} - \hat{Y}\|^2}{\|\bar{Y}\|^2},
\]
(21)
where \(\|\cdot\|_F\) symbolizes the Frobenius norm [6, p.55] of a matrix. In this simulation under different channel matrices, the better SVD algorithm with better estimation has lower NMSE. In other words, if we have a perfect estimation of the SVD results, the value of NMSE would be zero.

In Fig. 3, we use \(4 \times 4\) complex channel matrices and evaluate the NMSE performance. Deriving each right singular vector, we use different iteration number and wordlength precision for computation. The proposed SVD algorithm needs only 1–4 iterations for the performance to converge. In Fig.4, we set the channel matrices to be \(3 \times 2\) asymmetric complex matrices and the SVD algorithm still works. The convergence speed in these cases is even faster than that in the \(4 \times 4\) cases due to the rank deficiency, and only 1–2 iterations needed of deriving each right singular vectors.

In Fig. 5, BPSK-modulated signals are exploited to evaluate the BER performance versus different signal-to-noise ratios (SNR) and different wordlength precisions. The channel matrices are set to be \(4 \times 4\) complex matrices. The unitary matrix \(\bar{V}\) is feedback to transmitter for precoding and the other unitary matrix \(\bar{U}\) is used for postcoding at the receiver. As shown in Fig. 5, the proposed SVD algorithm does not have acceptable performance with the wordlength of 4 and 6 bits, and it works well with 8 and 10 bit precision. In the situation of lower SNR, the BER performances with 8 and 10 bits are almost as good as that with the floating-point precision. In the situation of higher SNR, the algorithm with 10 bit precision still works well.

V. CONCLUSIONS

In this paper, a SVD algorithm with fast convergence speed is proposed. The algorithm can obtain the SVD results of the complex MIMO channel matrices in at most four iterations. The fast convergence speed makes this algorithm suitable for the MIMO channels with short coherent time. Firstly, we find out the directions of the singular vectors in this algorithm. So we can apply the proposed adaptive binary shift mechanism to the computations in iterations without losing much information of singular vectors. Moreover, a SVD engine using this algorithm with little hardware overhead can not only decompose the \(N_t \times N_r\) channel matrices, but also handle smaller matrices. The complexity in every iterative computation is high, but the total computation complexity is low owing to the fast convergence speed.

REFERENCES